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Modular Functions Arising From Some Finite Groups

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Abstract. In [2] Conway and Norton have assigned a "Thompson series" of the form

$$q^{-1} + H_1 q + H_2 q^2 + \dots$$

to each element m of the Fischer-Griess "Monster" group M and conjectured that these functions are Hauptmoduls for certain genus-zero modular groups. We have found, for a large number of values of N, all the genus-zero groups between $\Gamma_0(N)$ and $PSL(2, R)$ that have Hauptmoduls of the above form, and this provides the necessary verification that the series assigned in [2] to particular elements of M really are such Hauptmoduls. (Atkin and Fong [1] have recently verified that $H_n(m)$ really is a character of M for all n.) We compute Thompson series for various finite groups and discuss the differences between these groups and M. We find that the resulting Thompson series are all Hauptmoduls for suitable genus-zero subgroups of $PSL(2, R)$.

1. Summary. Some remarkable connections between the Fischer-Griess "Monster" group M and modular functions have recently been reported in [2] and [5]. It has been noticed that the coefficients in the q-series for j

$$j(z) = \sum_{n=-1}^{\infty} c_n q^n = q^{-1} + 744 + 196884q + 21493760q^2 + \dots,$$

where j is the fundamental modular function on $\Gamma = PSL(2, Z)$ and $q = e^{2\pi i z}$, are linear combinations of the character degrees d_n of M. By considering $J = j - 744$, i.e., disregarding the constant term, and replacing the coefficients c_n in the q-series for J by the corresponding representations of M, one obtains a formal power series [5]

$$H_{-1}q^{-1} + 0 + H_1 q + H_2 q^2 + H_3 q^3 + \dots,$$

where $q = e^{2\pi i z}$ and H_n is a representation of degree c_n known as a head representation. Replacing H_n by its character value $H_n(m)$ for various elements $m \in M$, we obtain other functions [5]:

$$T_m(z) = q^{-1} + H_1(m)q + H_2(m)q^2 + H_3(m)q^3 + \dots,$$

which are called the Thompson series in [2]. The $H_i(m)$ are called head characters. Conway and Norton [2] have computed the functions $T_m(z)$ for all elements m of M, and they conjectured that, for every $m \in M$, $T_m(z)$ is a Hauptmodul for a group $F(m)$ between $\Gamma_0(N)$ for some N and its normalizer in $PSL(2, R)$, i.e., $T_m(z)$ generates a genus-zero function field invariant under $F(m)$. In this paper we describe certain related calculations. The detailed working can be found in [4]. We have explicitly verified that the modular functions assigned to various m by

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Conway and Norton [2] are actually Hauptmoduls for the groups they mention. Atkin and Fong [1] have recently verified that $H_n(m)$ really is a character of M for all n.

We have also considered other finite groups, usually derived from centralizers of elements of M, and computed their head character tables. In particular, we consider

$.0 = 2.(.1)$	.1 is the Conway simple group
E	Thompson's simple group
3.2.Suz	Suz is Suzuki's sporadic simple group
2.HJ	HJ is the Hall-Janko simple group
F	Harada-Norton simple group
$2.A_7$	Schur double cover of A_7
H	Held's simple group
M_{12}	Mathieu simple group

for which we have

elements of M	centralizer in M	
$2B$	$2^{1+24} \cdot G$	$G = .1$
$3B$	$3^{1+12} \cdot G$	$G = 2.\text{Suz}$
$3C$	$3 \times E$	
$5A$	$5 \times F$	
$5B$	$5^{1+6} \cdot G$	$G = 2.\text{HJ}$
$7A$	$7 \times H$	
$7B$	$7^{1+4} \cdot G$	$G = 2.A_7$
$11A$	$11 \times M_{12}$	

2. Notation and Terminology. As usual, for a positive integer N, we define

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv d \equiv 1 \pmod{N}, b \equiv c \equiv 0 \pmod{N} \right\}$$

and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}.$$

We also define

$$\Gamma_0(n\colon h) = \left\{ \begin{pmatrix} a & b/h \\ cnh & d \end{pmatrix} : ad - bcn = 1 \right\},$$

which is a subgroup of $PSL(2, R)$ and is conjugate to $\Gamma_0(n)$. In [2] this group is denoted by $\Gamma_0(nh|h)$; however, we prefer to reserve this name to denote a subgroup of $\Gamma_0(n\colon h)$ of index h which has Hauptmodul $T_{nh|h}$ (see below). Thus, adapting the rest of the notation from [2], we always have the same name for the genus-zero subgroup of $PSL(2, R)$ and the corresponding Hauptmodul in the canonical form (i.e. beginning $q^{-1} + 0 + aq + \ldots$). Also, this notation provides a natural way of enumerating all discrete subgroups of $PSL(2, R)$ containing $\Gamma_0(N)$ for a given N; see [4].

Thus, we write

$\Gamma_0(n\colon h) + e, f, \ldots$	for $\langle \Gamma_0(n\colon h), w_e, w_f, \ldots \rangle$,
$\Gamma_0(n\colon h) +$	when all w_e for $\Gamma_0(n\colon h)$ are present,
$\Gamma_0(n\colon h) -$ or $\Gamma_0(n\colon h)$	when no $w_e \neq 1$ is present,

where

$$w_e = \left\{ \begin{pmatrix} ae & b/h \\ chn & de \end{pmatrix} : e \text{ divides } n \text{ exactly, and } ade^2 - bcn = e > 0 \right\}$$

is a single coset of $\Gamma_0(n\colon h)$. The w_e are called the Atkin-Lehner involutions for $\Gamma_0(n\colon h)$ [2]. The corresponding Hauptmoduls (when they exist) are denoted by $t_{n+e,f,\ldots}(hz)$, $t_{n+}(hz)$ and $t_{n-}(hz)$ or $t_n(hz)$, respectively. $\Gamma_0(n)$ and its Hauptmodul $t_n(z)$ are a particular case when $h = 1$.

$$T_{n+e,f,\ldots}(hz) = t_{n+e,f,\ldots}(hz) - \text{constant term}$$

is the canonical Hauptmodul for $\Gamma_0(n\colon h) + e, f, \ldots$.

If F is a subgroup of $\Gamma_0(n\colon h) + e, f, \ldots$ of index h, with Hauptmodul T and $(T(z))^h = T_n(hz) + K$, where K is a constant, we denote F by $\Gamma_0(nh \mid h) + e, f, \ldots$ and T by $T_{nh|h+e,f,\ldots}$.

In this work we also define

$$\Gamma_0\left(n\frac{f}{g}\right) = \begin{pmatrix} g & f \\ 0 & g \end{pmatrix} \Gamma_0(n) \begin{pmatrix} g & -f \\ 0 & g \end{pmatrix},$$

$$\Gamma_0\left(n\frac{f}{g}\right) + = \begin{pmatrix} g & f \\ 0 & g \end{pmatrix} \Gamma_0(n) + \begin{pmatrix} g & -f \\ 0 & g \end{pmatrix},$$

$$\Gamma_0\left(n\frac{f}{g}h\colon h\right) + = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \Gamma_0\left(n\frac{f}{g}\right) + \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix},$$

where g is such that $g \mid 24$ and $(g, f) = 1$. The corresponding functions are

$$t_{n(f/g)}(z) = e^{2\pi i f/g} t_n(z + f/g), \qquad t_{n(f/g)+}(z) = e^{2\pi i f/g} t_{n+}(z + f/g),$$

and $t_{n(f/g)+}(hz)$, respectively. These are used to label a wide class of groups and functions arising from various finite groups G considered in this paper.

From [2] we quote certain identities which are called there replication formulae:

$$\tfrac{1}{2}\{T^2 - T_{(2)}(2z)\} = \{H_2 q + H_4 q^2 + \ldots\} + H_1 \quad \text{(duplication)},$$

$$\tfrac{1}{3}\{T^3 - T_{(3)}(3z)\} = \{H_3 q + H_6 q^2 + \ldots\} + H_1 T + H_2 \quad \text{(triplication)},$$

$$\tfrac{1}{5}\{T^5 - T_{(5)}(5z)\} = \{H_5 q + H_{10} q^2 + \ldots\} + H_1 T^3 + H_2 T^2 + (H_3 - H_1^2)T + (H_4 - H_2 H_1),$$

where $T = T_m(z)$, $T_{(n)} = T_{m^n}$ and $H_r = H_r(m)$, for $m \in M$. Many identities are obtained by comparing powers of q.

3. Discussion of Calculations and Results. When computing their Thompson series, the groups mentioned above fall naturally into two classes. The first one consists of $G_2 = .0$, $G_3 = 3.2.\text{Suz}$, $G_5 = 2.\text{HJ}$ and $G_7 = 2.A_7$. Each of these groups has a central element -1 and two algebraically conjugate representations $[d_+]$ and $[d_-]$ of degree $d = 24/(p-1)$, where p is the order of the element of M whose centralizer involves G_p for $p = 2, 3, 5, 7$. If G is one of these groups, then for every $g \in G$ the Thompson series t_g is given by the formula

$$t_g(z) = q^{-1} \prod_{p \nmid n} (1 - \varepsilon_1 q^n)(1 - \varepsilon_2 q^n) \cdots (1 - \varepsilon_d q^n)$$
$$= q^{-1} + H_0(g) + H_1(g)q + H_2(g)q^2 + \ldots$$

($q = e^{2\pi i z}$), where the ε's are the eigenvalues of g in the representation $[d_{(n/p)}]$, and (n/p) is the Legendre symbol. The $H_i(g)$ is a generalized character of G, while it can be shown that $H_i(-g)$ is a proper character; see [2]. However, our calculations show that the replication order of t_g is always less than or equal to the replication order of t_{-g}, and so they cannot be interchanged. We also observe that, as was not the case for M, the constant term is significant. In fact, $H_0(-g)$ is the character of G corresponding to the representation $[d_+]$.

The second class includes E, F, H and M_{12}. To compute the Thompson series for $g \in G$, where G is one of the groups above, we have to find, by trial and error, linear combinations of irreducible characters of G that work, i.e., such that the resulting Thompson series can be identified as modular functions for certain discrete subgroups of $PSL(2, R)$. Of course, such linear combinations do not have to exist, and in fact it is quite amazing that they do. For these groups, $H_i(g)$ is a proper character of G and the constant term is immaterial.

Tables I–VIII give values of head characters for these groups, together with the decomposition of the $H_i(g)$ into the irreducible characters of G. We found that to every element $g \in G$, where G is one of the groups above, there corresponds a function

$$t_g(z) = q^{-1} + H_0(g) + H_1(g)q + H_2(g)q^2 + \ldots,$$

which can be identified as a Hauptmodul for a discrete subgroup $F = F(g)$ of $PSL(2, R)$ containing $\Gamma_0(N)$ for some N and such that $F_\infty = \langle \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) \rangle$, where F_∞ is the stabilizer of the cusp at $z = i\infty$. However, $F(g)$ is not necessarily contained in the normalizer of $\Gamma_0(N)$ in $PSL(2, R)$, as was the case for M. Tables I–VIII also include the corresponding $\Gamma_0(N)$ and the type t (i.e., the name of the fixing group and the corresponding Hauptmodul) of $t_g(z)$ for every $g \in G$.

Some observations resulting from this work, in particular some necessary alterations in the replication formulae and the more general form of fixing group $F(g)$, have already been reported upon in [2].

Let G be one of the groups considered above. Let p be the order of the element of M from whose centralizer G was derived. If $(n/p) = -1$, $H_n(g)$ are rational for all $g \in G$, and the n-tuplication formulae are used with algebraic conjugation.

If $g \in G$ of order s such that $(s, p) = 1$, then its Thompson series T_g is the same as T_m for some $m \in M$, i.e., we can obtain new functions only from the elements of G whose order is divisible by p.

Consider $T = T_g$, where $g \in M$ or $g \in G$ for one of the groups G discussed above. The replication formulae [2] define a function $T_{(n)}$, called the n-tuplicate of T. If $T = T_m$ for $m \in M$,

$$T_{(n)} = T_{m^n}.$$

For an arbitrary Thompson series T, we say that T has replication order n if

$$T_{(n)} = J.$$

We note that J is a self-replicating function, and it assumes the role of the identity.

Let G and p be as above and let $T = T_g$ for $g \in G$. If $(n, p) = 1$,

$$T_{(n)} = T_{g^n}.$$

Hence, if $g \in G$ and $o(g) = s$ such that $(s, p) = 1$,

$$T_{(s)} = T_{1A(G)},$$

where $1A(G)$ is the identity element of G, and T has replication order ps. On the other hand, if $(n, p) \neq 1$,

$$T_{(n)} = T_m,$$

for some $m \in M$. For example, we found in F that T_{5A} has replication order 5, T_{10B} has replication order 10, T_{15A} has replication order 15, T_{20C} has replication order 20 and T_{25A}, T_{25B} have replication order 25, where $5A$, $10B$, $15A$, $20C$, $25A$, and $25B$ are elements of order 5, 10, 15, 20, and 25, respectively.

We would like to note that since $G = 3.2.\mathrm{Suz}$ contains a central element ω of order 3, for every $t_g(z)$ given in Table II we also have

$$t_{\omega g}(z) = \omega t_g(z + 1/3)$$

and

$$t_{\bar{\omega} g}(z) = \bar{\omega} t_g(z + 2/3) = t^*_{\omega g}(z),$$

where * denotes algebraic conjugation.

In our calculations we used the character tables from [3].

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TABLE I

Values of head characters for .0

		−1A +1A	−2A +2A	±2B	±2C	−3A +3A	−3B +3B	−3C +3C
H_{-1}	1	1	1	1	1	1	1	1
H_0	24	±24	∓8	0	0	∓12	±6	∓3
H_1	276	276	20	12	−12	78	15	6
H_2	24+2024	±2048	0	0	0	∓364	±32	∓4
H_3	1+276+299+1771+8855	11202	−62	66	66	1365	87	3
H_4	$24^2+2024^2+4576+40480$	±49152	0	0	0	∓4380	±192	±12
H_5	$1+276^4+299+1771^2+8855+37674^2+94875$	184024	216	232	−232	12520	343	−8
H_6		±614400	0	0	0	∓32772	±672	∓12
H_7		1881471	−641	639	639	80094	1290	30
H_8		±5373952	0	0	0	∓185276	±2176	∓20
H_9		14478180	1636	1596	−1596	409578	3705	30
H_{10}		±37122048	0	0	0	∓871272	±6336	±72
$\Gamma_o(N)$		4 2	4 4	16	8	12 6	12 6	12 6
t		4+ 2−	4− 4−	8|2+	4|2−	$6\frac{1}{2}$+6 6+6	12+ 6+3	12+4 6−

TABLE I (*continued*)

	−3D +3D	−4A +4A	±4B	−4C +4C	±4D	±4E	±4F	−5A +5A	−5B +5B	−5C +5C	−6A +6A	±6B	−6C +6C	−6D +6D
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	0	∓8	0	4	0	0	0	∓6	±4	∓1	±4	0	±4	∓5
H_1	0	36	4	4	−4	0	0	21	6	1	14	−6	5	14
H_2	±8	∓128	0	0	0	0	0	∓62	±8	∓2	±36	0	0	∓36
H_3	0	386	2	2	2	6	−6	162	17	2	85	21	−5	85
H_4	0	∓1024	0	0	0	0	0	∓378	±32	±2	±180	0	0	∓180
H_5	28	2488	−8	−8	8	0	0	819	54	1	360	−56	9	360
H_6	0	∓5632	0	0	0	0	0	∓1680	±80	0	±684	0	0	∓684
H_7	0	12031	−1	−1	−1	15	15	3276	116	4	1246	126	−14	1246
H_8	±64	∓24576	0	0	0	0	0	∓6138	±192	±2	±2196	0	0	∓2196
H_9	0	48308	20	20	20	0	0	11145	290	5	3754	−258	19	3754
H_{10}	0	∓91904	0	0	0	0	0	∓19662	±408	∓2	±6264	0	0	∓6264
$\Gamma_0(N)$	36 18	8 8	8	8 8	16	64	32	20 10	20 10	20 10	12 12	48	12 12	12 12
t	12|3+ 6|3	$8\frac{1}{2}+$ 8+	8−	8− 8−	8|2−	16|4+	8|4−	$10\frac{1}{2}+10$ 10+10	20+ 10+5	20+4 10−	12+12 $12\frac{1}{2}+12$	$12\frac{1}{2}$|2+6	12|2+$\bar{3}$ 12|2+$\bar{3}$	$12\frac{1}{2}+12$ 12+12

TABLE I (*continued*)

	-6E +6E	-6F +6F	±6G	±6H	±6I	-7A +7A	-7B +7B	±8A	±8B	-8C +8C	±8D	-8E +8E	±8F	-9A +9A	-9B +9B
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	∓2	∓1	0	0	0	∓4	±3	0	0	∓4	0	∓2	0	∓3	0
H_1	-1	2	-3	0	0	10	3	4	-4	8	0	0	0	6	0
H_2	0	0	0	0	0	∓24	±4	0	0	∓16	0	0	0	∓13	∓1
H_3	7	1	3	0	0	51	9	10	10	34	2	2	-2	24	0
H_4	0	0	0	0	0	∓100	±12	0	0	∓64	0	0	0	∓42	0
H_5	-9	0	-7	4	-4	190	15	24	-24	112	0	0	0	73	1
H_6	0	0	0	0	0	∓340	±24	0	0	∓192	0	0	0	∓120	0
H_7	10	-2	18	0	0	585	39	47	47	319	-1	-1	-1	192	0
H_8	0	0	0	0	0	∓984	±52	0	0	∓512	0	0	0	∓299	±1
H_9	-23	-2	-21	0	0	1606	66	84	-84	808	0	0	0	456	0
H_{10}	0	0	0	0	0	∓2564	±96	0	0	∓1248	0	0	0	∓684	0
$\Gamma_o(N)$	12 12	12 12	24	114	72	28 14	28 14	32	32	16 16	16	16 16	64	36 18	36 18
t	12+3 12+3	12- 12-	12|2+3	24|6+	12|6	$14\frac{1}{2}$+14 14+14	28+ 14+7	16|2+	$16\frac{1}{2}$|2+	$16\frac{1}{2}$+ 16+	16-	16- 16-	$16\frac{1}{8}$-	$18\frac{1}{2}$+18 18+18	36+4 18-

TABLE I (*continued*)

	−9C +9C	−10A +10A	±10B	±10C	−10D +10D	−10E +10E	±10F	−11A +11A	−12A +12A	±12B	±12C	−12D +12D	−12E +12E
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	±3	±2	0	0	±2	∓3	0	±2	±4	0	0	±1	∓2
H_1	3	5	−3	2	0	5	−2	1	6	−2	2	0	3
H_2	±2	±10	0	0	0	∓10	0	±2	±4	0	0	∓2	∓8
H_3	3	18	6	1	3	18	1	4	5	5	5	−1	11
H_4	±6	±30	0	0	0	∓30	0	±4	±20	0	0	±2	∓16
H_5	10	51	−13	2	−4	51	−2	5	40	−8	8	4	31
H_6	±12	±80	0	0	0	∓80	0	±6	±44	0	0	±2	∓40
H_7	15	124	24	4	4	124	4	9	46	14	14	−2	58
H_8	±22	±190	0	0	0	∓190	0	±12	±84	0	0	∓6	∓96
H_9	30	281	−39	6	−4	281	−6	13	146	−22	22	−4	125
H_{10}	±36	±410	0	0	0	∓410	0	±18	±184	0	0	±4	∓176
$\Gamma_o(N)$	36 18	20 20	80	80	40 40	20 20	40	44 22	24 24	24	48	24 24	24 24
t	36+ 18+9	20+20 $20\frac{1}{2}$+20	$20\frac{1}{2}$|2+10	40|2+	20|2+5 20|2+5	$20\frac{1}{2}$+20 20+20	20|2+5	44+ 22+11		$24\frac{1}{2}$|2 +12	24|2+12	$24\frac{1}{2}$+8 24+8	$24\frac{1}{2}$+ 24+

TABLE I (*continued*)

	±12F	±12G	−12H +12H	−12I +12I	±12J	−12K +12K	±12L	±12M	−13A +13A	±14A	−14B +14B	−15A +15A	−15B +15B
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	0	0	∓1	±2	0	∓3	0	0	2	0	∓1	±3	∓2
H_1	0	1	−2	1	−1	4	0	0	3	−2	−1	3	3
H_2	0	0	0	0	0	∓6	0	0	6	0	0	±1	∓4
H_3	−3	−1	5	−1	−1	11	0	0	9	3	1	0	5
H_4	0	0	0	0	0	∓18	0	0	14	0	0	0	∓10
H_5	0	1	−8	1	−1	28	0	0	22	−6	−1	0	15
H_6	0	0	0	0	0	∓42	0	0	32	0	0	±3	∓22
H_7	6	2	14	2	2	62	0	0	46	9	3	9	29
H_8	0	0	0	0	0	∓90	0	0	66	0	0	±9	∓36
H_9	0	−1	−22	−1	1	128	0	0	93	−14	−2	3	53
H_{10}	0	0	0	0	0	∓180	0	0	128	0	0	±3	∓72
$\Gamma_o(N)$	192	24	24 24	24 24	48	24 24	576	288	52 26	112	28 28	60 30	60 30
t	$24\frac{1}{2}$|4+6	24|2+3	$24\frac{1}{2}$|2+12 $24\frac{1}{2}$|2+12	24|2+3 24|2+3	24|2+3	$24\frac{1}{2}$+24 24+24	48|12+	24|12−	$26\frac{1}{2}$+26 26+26	$28\frac{1}{2}$|2+14	28+7 28+7	$30\frac{1}{2}$+6, 10,15 30+6, 10,15	$30\frac{1}{2}$+5, 6,30 30+5, 6,30

TABLE I (*continued*)

	−15C +15C	−15D +15D	−15E +15E	±16A	−16B +16B	−18A +18A	−18B +18B	−18C +18C	−20A +20A	±20B	−20C +20C	−21A +21A	−21B +21B
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	0	±1	±2	0	∓2	±1	∓2	±1	±2	0	±1	±2	∓1
H_1	0	0	1	−2	2	2	2	1	1	0	−1	1	1
H_2	∓2	±2	±1	0	∓4	±3	∓3	0	±2	0	0	0	∓3
H_3	0	2	2	2	6	4	4	1	6	−1	2	0	3
H_4	0	±2	±2	0	∓8	±6	∓6	0	±6	0	0	±2	∓4
H_5	3	3	2	−4	12	9	9	0	3	0	−3	4	7
H_6	0	±2	±3	0	∓16	±12	∓12	0	±8	0	0	±2	∓7
H_7	0	5	5	7	23	16	16	1	16	4	4	0	9
H_8	∓6	±6	±5	0	∓32	±21	∓21	0	±14	0	0	0	∓15
H_9	0	5	5	−10	42	28	28	−2	13	0	−5	1	16
H_{10}	0	±6	±7	0	∓56	±36	∓36	0	±26	0	0	±4	∓20
$\Gamma_o(N)$	180 90	60 30	60 30	64	32 32	36 36	36 36	36 36	40 40	160	40 40	84 42	84 42
t	$30\frac{1}{2}\|3$ +10	$30\frac{1}{2}+3$, 5,15	$30\frac{1}{2}+15$	$32\frac{1}{2}\|2+$	$32\frac{1}{2}+$	36+36	$36\frac{1}{2}+36$	36|2+9		40|4+5	$40\frac{1}{2}\|2+20$	$42\frac{1}{2}+6$, 14,21	$42\frac{1}{2}+3$, 14,42
	30|3 +10	30+3, 5,15	30+15		32+	$36\frac{1}{2}+36$	36+36	36|2+9			$40\frac{1}{2}\|2+20$	42+6, 14,21	42+3, 14,42

TABLE I (*continued*)

	−21C +21C	$\pm 2\overset{*}{2}$A	$-2\overset{*}{3}$A +23A	±24A	−24B +24B	±24C	±24D	±24E	−24F +24F	−28A +28A	$\pm 2\overset{*}{8}$B	−30A +30A	±30B
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	0	0	±1	0	±2	0	0	0	±1	∓1	0	∓1	0
H_1	0	−1	0	−2	2	1	0	−1	0	1	0	1	−1
H_2	±1	0	±1	0	±2	0	0	0	0	∓2	0	±1	0
H_3	0	0	1	1	1	1	1	1	−1	1	−1	0	1
H_4	0	0	±1	0	±2	0	0	0	0	∓2	0	0	0
H_5	0	−1	1	0	4	3	0	−3	0	3	0	0	−1
H_6	0	0	±1	0	±6	0	0	0	0	∓4	0	∓1	0
H_7	0	1	2	2	10	2	2	2	2	5	1	1	1
H_8	±1	0	±2	0	±10	0	0	0	0	∓6	0	±1	0
H_9	0	−1	2	−6	10	3	0	−3	0	8	0	1	−3
H_{10}	0	0	±2	0	±12	0	0	0	0	∓8	0	∓1	0
$\Gamma_o(N)$	252 126	88	92 46	96	48 48	96	192	96	48 48	56 56	448	60 60	240
t	$42\frac{1}{2}\|3+7$ $42\|3+7$	$44\|2+11$	$46\frac{1}{2}+23$ $46+23$	$(48\|2)$		$48\|2+$	$48\|4+12$	$48\frac{1}{2}\|2+$	$48\frac{1}{2}\|4+12$ $48\frac{1}{2}\|4+12$	$56\frac{1}{2}+$ $56+$	$56\frac{1}{2}\|4+14$	$60+12$, 15, 20 $60\frac{1}{2}+12$ 15, 20	$60\frac{1}{2}\|2+5$, 6, 30

TABLE II

Values of head characters for 3.2.Suz ($\theta = \sqrt{-3}$)

			1A		2A		± 2B	3A	
			+	−	+	−		+	−
H_{-1}	1		1	1	1	1	1	1	1
H_0		12*	−12	12	4	−4	0	6	−6
H_1	66	12	54	78	6	−2	6	27	15
H_2	1+143	220	−76	364	4	28	0	86	−14
H_3	$(66^*)^2+780^*$	$(12^*)^2+780^*$	−243	1365	−3	−27	21	243	−21
H_4	$66^2+78+429+2145$	12^3+780^2	1188	4380	−12	−52	0	594	78
H_5	$1^4+143^4+780^2+3432$	$220^4+572a+572b+4928$	−1384	12520	−8	136	56	1370	−62
H_6			−2916	32772	12	−108	0	2916	−132
H_7			11934	80094	30	−162	126	5967	399
H_8			−11580	185276	20	620	0	11586	−322
H_9			−21870	409578	−30	−486	258	21870	−426
H_{10}			79704	871272	−72	−760	0	39852	1332
$\Gamma_0(N)$			3	6	6	6	24	9	18
t			3	6+6	6	6+2	12|2+6	9+	

TABLE II (*continued*)

	3B		3C		4A		± 4B	± 4C	± 4D	5A		5B	
	+	−	+	−	+	−				+	−	+	−
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	-3	3	0	0	4	-4	0	0	0	3	-3	-2	2
H_1	0	6	0	0	14	6	2	-2	0	9	3	-1	3
H_2	5	13	-4	4	36	-4	0	0	0	19	-1	4	4
H_3	0	24	0	0	85	-3	1	-3	3	42	0	-3	5
H_4	0	42	0	0	180	12	0	0	0	78	0	-2	10
H_5	-7	73	2	10	360	-8	0	8	0	146	0	11	15
H_6	0	120	0	0	684	-12	0	0	0	249	-3	-6	22
H_7	0	192	0	0	1246	30	-2	-2	6	429	9	-11	29
H_8	3	299	12	20	2196	-20	0	0	0	695	-9	20	36
H_9	0	456	0	0	3754	-30	-2	-6	0	1125	3	-15	53
H_{10}	0	684	0	0	6264	72	0	0	0	1749	-3	-16	72
$\Gamma_0(N)$	9	18	27	54	12	12	12	24	96	15	30	15	30
t	9	18+18	9|3	18|3 +6	12+12	12+4	12	24|2+2	24|4+6	15+15	30+6, 10,15	15+5	30+5,6,30

TABLE II (*continued*)

	6A +	6A −	6C 6B +	6C 6B −	6D +	6D −	± 6E	7A +	7A −	± 8A	8B +	8B −	± 8C
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	−2	2	$-1+2\omega$	$1-2\omega$	1	−1	0	2	−2	0	−2	2	0
H_1	3	7	$6\bar{\omega}$	-2ω	0	−2	0	5	1	2	0	4	0
H_2	−2	10	13	1	1	1	0	8	0	0	2	6	0
H_3	3	27	24ω	0	0	0	0	16	0	5	−1	11	−1
H_4	−6	38	$42\bar{\omega}$	2ω	0	2	0	26	−2	0	−2	18	0
H_5	10	82	73	1	1	1	2	44	4	8	4	28	0
H_6	−12	108	120ω	0	0	0	0	66	−2	0	−2	42	0
H_7	15	207	$192\bar{\omega}$	0	0	0	0	104	0	14	−2	62	−2
H_8	−22	278	299	−1	−1	−1	0	152	0	0	6	90	0
H_9	30	486	456ω	0	0	0	0	229	1	22	−4	128	0
H_{10}	−36	644	$684\bar{\omega}$	-4ω	0	−4	0	324	−4	0	−4	180	0
$\Gamma_0(N)$	18	18	18	18	18	18	216	21	42	48	24	24	96
t	18+9	18+	$18\frac{1}{3}+18$	$18\frac{2}{3}+2$	18	18+2	36|6 +6	21+21	42+6, 14,21	24|2+12	24+8	24	24|4+2

TABLE II (*continued*)

	9B 9A +	9B 9A −	10A +	10A −	± 10B	11A +	11A −	12A +	12A −	12B +	12B −	± 12C
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1
H_0	$-\theta$	θ	−1	1	0	−1	1	−2	2	1	−1	0
H_1	3ω	$-3/2-\theta/2$	1	3	1	−1	1	−1	3	2	0	1
H_2	5	1	−1	3	0	1	1	6	2	3	−1	0
H_3	$9\bar{\omega}$	$-3/2+\theta/2$	2	8	1	−1	1	−5	3	4	0	3
H_4	12ω	-2θ	−2	8	0	0	2	−6	6	6	0	0
H_5	20	4	2	16	1	2	2	18	10	9	1	2
H_6	$27\bar{\omega}$	$-3/2+5\theta/2$	−3	17	0	−1	3	−12	12	12	0	0
H_7	42ω	$-3-\theta$	5	33	1	−1	3	−17	15	16	0	7
H_8	57	5	−5	35	0	3	3	42	22	21	1	0
H_9	$81\bar{\omega}$	$9-10\theta$	5	59	3	−2	4	−22	30	28	0	6
H_{10}	108ω	$-6-6\theta$	−7	65	0	−2	6	−36	36	36	0	0
$\Gamma_0(N)$	27	54	30	30	120	33	66	36	36	36	36	72
t	$27\frac{2}{3}+$		30+15	30+2, 15,30	60|2+5, 6,30	33+11	66+6, 11,66		36+	36+36	36+4	36|2+

TABLE II (*continued*)

	± 12D	12E +	12E −	13B 13A +	13B 13A −	± 14A	15B 15A +	15B 15A −	15C +	15C −	18B 18A +	18B 18A −
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1
H_0	0	θ	θ	1	−1	0	0	0	1	−1	1	−1
H_1	0	$2\bar{\omega}$	2ω	2	0	−1	0	0	2	0	$3/2+\theta/2$	ω
H_2	0	3	3	2	0	0	1	−1	1	1	1	1
H_3	0	4ω	$4\bar{\omega}$	4	0	0	0	0	3	−1	$3/2-\theta/2$	$3\bar{\omega}$
H_4	0	$6\bar{\omega}$	6ω	5	−1	0	0	0	4	−2	$3-\theta$	2ω
H_5	0	9	9	7	1	0	2	0	5	3	4	4
H_6	0	12ω	$12\bar{\omega}$	9	−1	0	0	0	6	−2	$5/2+\theta/2$	$3\bar{\omega}$
H_7	0	$16\bar{\omega}$	16ω	13	1	0	0	0	7	−1	$1-\theta$	6ω
H_8	0	21	21	16	0	0	2	0	11	3	5	5
H_9	0	28ω	$28\bar{\omega}$	22	0	−1	0	0	15	−1	7	$9\bar{\omega}$
H_{10}	0	$36\bar{\omega}$	36ω	27	−1	0	0	0	17	−3	12	8ω
$\Gamma_0(N)$	864	36	36	39	78	168	135	270	45	90	54	54
t	72|12	$36\frac{1}{3}+36$	$36\frac{2}{3}+36$	39+39	78+6, 26,39	84|2+6 14,21	45|3 +15	90|3+6 10,15	45+			$54\frac{2}{3}+$

TABLE II (*continued*)

	20A +	20A −	21B 21A +	21B 21A −	24A +	24A −
H_{-1}	1	1	1	1	1	1
H_0	-1	1	-1	1	0	0
H_1	-1	1	-1	1	-1	-1
H_2	1	1	2	0	0	0
H_3	0	2	-2	0	-1	-1
H_4	0	2	-1	1	0	0
H_5	0	2	5	1	2	2
H_6	-1	3	-3	1	0	0
H_7	1	5	-4	0	-1	-1
H_8	1	5	8	0	0	0
H_9	-1	5	-5	1	-1	-1
H_{10}	-1	7	-6	2	0	0
$\Gamma_o(N)$	60	60	63	126	144	144
t	60+12, 15,20	60+4, 15,60			72|2	72|2

TABLE III

Values of head characters for E

		1A	2A	3A	3B	3C	4A	4B	5A	6A	6B
H_{-1}	1	1	1	1	1	1	1	1	1	1	1
H_2	248	248	-8	14	5	-4	8	0	-2	4	-2
H_5	1+4123	4124	28	65	-7	2	28	-4	-1	10	1
H_8	1+4123+30628	34572	-64	156	3	12	64	0	2	20	-4
H_{11}	$1+248^2+4123+61256+147250$	213126	134	456	15	-21	134	6	1	35	8
H_{14}		1057504	-288	1066	-32	4	288	0	4	60	-6
$\Gamma_0(N)$		9	18	27	9	27	36	72	45	54	54
t		3|3	6|3	9|3+	9-	9|3	12|3+	12|6	15|3	18|3 +6	18|3+3

TABLE III (*continued*)

	6C	7A	8A	8B	9A	9B	9C	10A	12B 12A	12C	12D	13A	14A	15B 15A
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1	1
H_2	1	3	0	0	5	-4	2	2	2	-1	0	1	-1	1
H_5	1	8	4	0	-7	2	5	3	1	1	2	3	0	2
H_8	-1	11	0	0	3	12	6	6	4	1	0	3	-1	2
H_{11}	-1	25	6	-2	15	-21	12	9	8	-1	3	4	1	4
H_{14}	0	35	0	0	-32	4	16	12	6	0	0	6	-1	4
$\Gamma_0(N)$	18	63	144	288	9	27	81	90	108	36	216	117	126	135
t	18-	21|3+	24|6+	24|12-	9-	9|3	27|3+	30|3+10	36|3+	36+4	36|6+6	39|3+	42|3+7	45|3+15

TABLE III (*continued*)

	18A	18B	19A	20A	21A	24B 24A	24D 24C	27A	27C 27B	28A	30B 30A	31B 31A	36A
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1
H_2	1	-2	1	0	0	0	0	2	-1	1	-1	0	-1
H_5	1	1	1	1	2	1	0	-1	2	0	0	1	1
H_8	-1	2	1	0	2	0	0	0	3	1	0	1	1
H_{11}	-1	-4	3	1	1	0	1	3	0	1	0	1	-1
H_{14}	0	0	2	0	2	0	0	-2	1	1	0	1	0
$\Gamma_0(N)$	18	162	171	360	189	432	864	243	243	252	270	279	36
t	18–	(54|3)	57|3+	60|6 +10	63|3+	72|6+	72|12 +6	(81|3)	(81|3)	84|3+	90|3+6 10,15	93|3+	36+4

TABLE III (*continued*)

	36C 36B	39B 39A
H_{-1}	1	1
H_2	−1	1
H_5	1	0
H_8	1	0
H_{11}	−1	1
H_{14}	0	0
$\Gamma_o(N)$	36	351
t	36+4	117|3+

TABLE IV

Values of head characters for 2.HJ ($\varphi = \sqrt{5}$)

			1A +	1A −	2A +	2A −	± 2B	3A +	3A −	3B +	3B −	4A +	4A −
H_{-1}	1		1	1	1	1	1	1	1	1	1	1	1
H_0		6a	-6	6	2	-2	0	3	-3	0	0	-2	2
H_1	1+14a	6b	9	21	1	-3	3	9	3	0	0	1	5
H_2	36	6a+6b+14	10	62	2	6	0	19	-1	-2	2	-2	10
H_3	1+14a+14b+36	6a+6b+84	-30	162	2	2	6	42	0	0	0	2	18
H_4			6	378	-2	2	0	78	0	0	0	2	30
H_5			-25	819	-1	-5	13	146	0	-1	3	-1	51
H_6			96	1680	0	-16	0	249	-3	0	0	0	80
H_7			60	3276	-4	12	24	429	9	0	0	-4	124
H_8			-250	6138	-2	2	0	695	-9	2	6	2	190
H_9			45	11145	5	17	39	1125	3	0	0	5	281
H_{10}			-150	19662	2	-10	0	1749	-3	0	0	-2	410
$\Gamma_0(N)$			5	10	10	10	40	15	30	45	90	20	20
t			5	10+10	10	10+2	20|2+10	15+15	30+6, 10,15	15|3	30|3 +10	20+4	20+20

Table IV (*continued*)

	5B 5A +	5B 5A −	5D 5C +	5D 5C −	6A +	6A −	± 6B	7A +	7A −	8A +	8A −
H_{-1}	1	1	1	1	1	1	1	1	1	1	1
H_0	$-1+\varphi$	$1-\varphi$	$3/2+\varphi/2$	$-3/2-\varphi/2$	−1	1	0	1	−1	0	0
H_1	$3/2-5\varphi/2$	$7/2-\varphi/2$	4	$1+\varphi$	1	3	0	2	0	1	1
H_2	−10	2	5	−3	−1	3	0	3	−1	0	0
H_3	5	−3	10	2	2	8	0	5	1	2	2
H_4	$21+5\varphi$	$3+3\varphi$	16	$3-\varphi$	−2	8	0	6	0	0	0
H_5	$-25/2+25\varphi/2$	$3/2-3\varphi/2$	25	$-1-2\varphi$	2	16	1	10	0	3	3
H_6	$6-30\varphi$	$10-2\varphi$	36	4φ	−3	17	0	12	0	0	0
H_7	−85	11	55	−9	5	33	0	18	0	4	4
H_8	50	−2	75	3	−5	35	0	23	−1	0	0
H_9	$285/2+65\varphi/2$	$25/2+13\varphi/2$	110	$15-3\varphi$	5	59	0	31	1	5	5
H_{10}	$-75+75\varphi$	$7-7\varphi$	150	$-8-6\varphi$	−7	65	0	39	−1	0	0
$\Gamma_0(N)$	25	50	25	50	30	30	360	35	70	80	80
t			25+		30+15	30+2, 15,30	60|6+10	35+35	70+10, 14,35	40|2+20	

TABLE IV (*continued*)

	± 10B ± 10A	10D 10C +	10D 10C −	12A +	12A −	15B 15A +	15B 15A −
H_{-1}	1	1	1	1	1	1	1
H_0	0	$-1/2+\varphi/2$	$1/2-\varphi/2$	1	−1	$1/2-\varphi/2$	$-1/2+\varphi/2$
H_1	$1/2+\varphi/2$	$1-\varphi$	2	1	−1	$3/2+\varphi/2$	$1/2-\varphi/2$
H_2	0	−3	1	1	1	−1	−1
H_3	1	2	2	2	0	2	0
H_4	0	$3+\varphi$	2	2	0	$3-\varphi$	0
H_5	$1/2-\varphi/2$	$-1+2\varphi$	5	2	0	$1-\varphi$	0
H_6	0	-4φ	4	3	−1	$3/2+3\varphi/2$	$-1/2-\varphi/2$
H_7	−1	−9	7	5	1	−1	−1
H_8	0	3	7	5	1	5	1
H_9	$3/2-\varphi/2$	$15+3\varphi$	12	5	−1	$15/2-\varphi/2$	$1/2+\varphi/2$
H_{10}	0	$-8+6\varphi$	10	7	−1	$3/2-3\varphi/2$	$-1/2+\varphi/2$
$\Gamma_0(N)$	100	50	50	60	60	75	150
t			50+	60+4, 15,60	60+12, 15,20		

TABLE V

Values of characters for F $(\varphi = \sqrt{5})$

		1A	2A	2B	3A	3B	4A	4B	4C	5A	5B
H_{-1}	1	1	1	1	1	1	1	1	1	1	1
H_0	*	*	*	*	*	*	*	*	*	*	*
H_1	1+133b	134	22	6	8	-1	6	2	-2	-6	9
H_2	760	760	56	-8	22	4	8	0	0	20	10
H_3	1+3344	3345	177	17	42	-3	17	9	1	15	-30
H_4	1+133a+3344+8778a	12256	352	-32	70	-2	32	0	0	36	6
H_5	1+133a+760+3344+35112a	39350	870	54	155	11	54	10	-2	0	-25
H_6		114096	1584	-80	246	-6	80	0	0	-84	96
H_7		307060	3412	116	421	-11	116	28	4	195	60
H_8		776000	5952	-192	722	20	192	0	0	100	-250
H_9		1867170	11442	290	1101	-15	290	30	-6	240	45
H_{10}		4298600	19240	-408	1730	-16	408	0	0	0	-150
$\Gamma_0(N)$		5	10	10	15	15	20	40	40	25	5
t		5+	10+	10+5	15+	15+5	20+	20|2+	20|2 +5		5

TABLE V (*continued*)

	5D 5C	5E	6A	6B	6C	7A	8A	8B	9A	10A	10B	10C	10E 10D
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	*	*	*	*	*	*	*	*	*	*	*	*	*
H_1	$3/2-5\varphi/2$	1	4	0	3	1	0	2	2	-3	2	1	$7/2-\varphi/2$
H_2	-10	4	2	-2	4	4	0	0	1	6	-4	2	2
H_3	5	5	6	2	5	6	1	1	3	2	7	2	-3
H_4	$21+5\varphi$	10	10	-2	10	6	0	0	4	2	12	-2	$3+3\varphi$
H_5	$-25/2+25\varphi/2$	16	15	3	15	10	0	2	5	-5	0	-1	$3/2-3\varphi/2$
H_6	$6-30\varphi$	25	18	-2	22	10	0	0	6	-16	4	0	$10-2\varphi$
H_7	-85	36	37	5	29	19	4	4	7	12	-13	-4	11
H_8	50	55	30	-6	36	22	0	0	11	2	12	-2	-2
H_9	$285/2+65\varphi/2$	75	57	5	53	32	0	6	15	17	32	5	$25/2+13\varphi/2$
H_{10}	$-75+75\varphi$	150	70	-6	72	40	0	0	17	-10	0	2	$7-7\varphi$
$\Gamma_0(N)$	25	25	30	30	30	35	160	80	45	10	50	10	50
t		25+	30+	30+3, 5,15	30+5, 6,30	35+	40|4+	40|2+	45+	10+2		10	

TABLE V (*continued*)

	10F	10H 10G	11A	12A	12B	12C	14A	15A	15C 15B	19B 19A	20B 20A	20C
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1
H_0	*	*	*	*	*	*	*	*	*	*	*	*
H_1	2	$1-\varphi$	2	2	0	1	1	3	$3/2+\varphi/2$	1	1	2
H_2	1	-3	1	0	2	0	0	2	-1	0	-2	0
H_3	2	2	1	0	2	1	2	-3	2	1	2	-1
H_4	2	$3+\varphi$	2	0	2	0	2	0	$3-\varphi$	1	2	0
H_5	5	$-1+2\varphi$	3	1	3	1	2	0	$1-\varphi$	1	-1	0
H_6	4	-4φ	4	0	2	0	2	6	$3/2+3\varphi/2$	1	0	0
H_7	7	-9	6	1	5	1	3	6	-1	1	-4	3
H_8	7	3	5	0	6	0	2	-8	5	2	2	0
H_9	12	$15+3\varphi$	8	3	5	3	4	6	$15/2-\varphi/2$	2	5	0
H_{10}	10	$-8+6\varphi$	9	0	6	0	4	0	$3/2-3\varphi/2$	2	-2	0
$\Gamma_0(N)$	50	50	55	120	60	120	70	75	75	95	20	100
t	50+		55+	60|2+	60+	60|2+5, 6,30	70+			95+	20+4	

TABLE V (*continued*)

	20E 20D	21A	22A	25B 25A	30A	30C 30B	35B 35A	40B 40A
H_{-1}	1	1	1	1	1	1	1	1
H_0	*	*	*	*	*	*	*	*
H_1	$1/2+\varphi/2$	1	0	$3/2+\varphi/2$	−1	$1/2-\varphi/2$	1	0
H_2	0	1	1	0	2	−1	−1	0
H_3	1	0	1	0	1	0	1	1
H_4	0	0	0	$1-\varphi$	0	0	1	0
H_5	$1/2-\varphi/2$	1	1	0	0	0	0	0
H_6	0	1	0	$1+\varphi$	−2	$-1/2-\varphi/2$	0	0
H_7	−1	1	2	0	2	−1	−1	4
H_8	0	1	1	0	0	1	2	0
H_9	$3/2-\varphi/2$	2	2	$5/2-3\varphi/2$	2	$1/2+\varphi/2$	2	0
H_{10}	0	1	1	0	0	$-1/2+\varphi/2$	0	0
$\Gamma_0(N)$	100	105	110	125	150	150	175	160
t		105+	110+					40|4+

TABLE VI

Values of characters for $2.A_7$ ($\theta = \sqrt{-7}$)

	1A +	1A −	± 2A	3A +	3A −	3B +	3B −	± 4A	5A +	5A −	± 6A	7B/7A +	7B/7A −
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	-4	4	0	2	-2	-1	1	0	1	-1	0	-1/2+θ/2	1/2-θ/2
H_1	2	10	2	5	1	-1	1	0	2	0	-1	-3/2+θ/2	-1/2-θ/2
H_2	8	24	0	8	0	-1	3	0	3	-1	0	-5/2-3θ/2	-1/2+θ/2
H_3	-5	51	3	16	0	1	3	1	5	1	0	2	2
H_4	-4	100	0	26	-2	2	4	0	6	0	0	3-θ	2
H_5	-10	190	6	44	4	-1	7	0	10	0	0	-3	1
H_6	12	340	0	66	-2	3	7	0	12	0	0	12	4
H_7	-7	585	9	104	0	-1	9	1	18	0	0	-7/2+7θ/2	0
H_8	8	984	0	152	0	-1	15	0	23	-1	0	-19/2+11/2	1/2-θ/2
H_9	46	1606	14	229	1	-2	16	0	31	1	-1	-27/2-19θ/2	-1/2-θ/2
H_{10}	-36	2564	0	324	-4	0	20	0	39	-1	0	6	2
$\Gamma_0(N)$	7	14	56	21	42	21	42	224	35	70	168	49	98
t	7	14+14	28|2+ 14	21+21	42+6, 14,21	21+3	42+3, 14,42	56|4+ 14	35+35	70+10, 14,35	84|2+6, 14,21		

TABLE VII

Values of head characters for H ($\theta = \sqrt{-7}$)

		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1
H_0	*	*	*	*	*	*	*	*	*	*	*	*
H_1	51a	51	11	3	6	0	3	3	-1	1	2	0
H_2	51b+153a	204	20	-4	6	3	0	4	0	4	2	-1
H_3	1+680	681	57	9	15	0	1	9	1	6	3	0
H_4	1+680+1275c	1956	92	-12	30	0	0	12	0	6	2	0
H_5	1+51a+51b+680+4352	5135	207	15	41	8	7	15	-1	10	9	0
H_6		12360	312	-24	66	0	0	24	0	10	6	0
H_7		28119	623	39	111	0	7	39	3	19	11	0
H_8		60572	932	-52	146	11	0	52	0	22	14	-1
H_9		125682	1674	66	222	0	18	66	-2	32	18	0
H_{10}		251040	2464	-96	336	0	0	96	0	40	16	0
$\Gamma_0(N)$		7	14	14	21	63	56	28	28	35	42	126
t		7+	14+	14+7	21+	21|3+	28|2+	28+	28+7	35+	42+	42|3+7

TABLE VII (*continued*)

	7B 7A	7C	7E 7D	8A	10A	12A	12B	14B 14A	14D 14C	15A
H_{-1}	1	1	1	1	1	1	1	1	1	1
H_0	*	*	*	*	*	*	*	*	*	*
H_1	$-3/2+3\theta/2$	2	$-3/2-\theta/2$	1	1	0	0	$1/2-\theta/2$	$-1/2+\theta/2$	1
H_2	$1-\theta$	8	$-5/2+3\theta/2$	2	0	0	1	$-1+\theta$	$-1/2-\theta/2$	1
H_3	9	-5	2	1	2	1	0	1	2	0
H_4	$3-3\theta$	-4	$3+\theta$	2	2	0	0	$1-\theta$	2	0
H_5	4	-10	-3	3	2	1	0	4	1	1
H_6	12	12	12	4	2	0	0	4	4	1
H_7	0	-7	$-7/2-7\theta/2$	5	3	1	0	0	0	1
H_8	$-13+13\theta$	8	$-19/2-11\theta/2$	6	2	0	1	$1-\theta$	$1/2+\theta/2$	1
H_9	$15/2-15\theta/2$	46	$-27/2+19\theta/2$	8	4	0	0	$-5/2+5\theta/2$	$-1/2+\theta/2$	2
H_{10}	48	-36	6	8	4	0	0	0	2	1
$\Gamma_0(N)$	49	7	49	56	70	168	252	98	98	105
t		7		56+	70+	84|2+	84|3+			105+

TABLE VII (*continued*)

	17B 17A	21B 21A	21D 21C	28B 28A
H_{-1}	1	1	1	1
H_0	*	*	*	*
H_1	0	−1	0	$-1/2+\theta/2$
H_2	0	−1	$-1/2+\theta/2$	0
H_3	1	1	0	1
H_4	1	2	0	0
H_5	1	−1	1	0
H_6	1	3	0	0
H_7	1	−1	0	0
H_8	1	−1	$1/2-\theta/2$	0
H_9	1	−2	0	$1/2-\theta/2$
H_{10}	1	0	0	0
$\Gamma_o(N)$	119	21	441	196
t	119+	21+3		

TABLE VIII

Values of head characters for M_{12}

$(b = -1/2 + \sqrt{-11}/2,\ b^* = -1/2 - \sqrt{-11}/2)$

		1A	2A	2B	3A	3B	4A	4B	5A
H_{-1}	1	1	1	1	1	1	1	1	1
H_0	*	*	*	*	*	*	*	*	*
H_1	1+16a	17	5	1	−1	2	1	1	2
H_2	1+45	46	6	−2	1	4	2	2	1
H_3	1+16a+45+54	116	16	4	−1	5	4	4	1
H_4	$1+16b^2+45+54+120$	252	20	−4	0	6	4	4	2
H_5	$1^2+16b+16a^2+45^2+54^2+66+99+120$	533	41	5	2	14	5	5	3
H_6		1034	50	−6	−1	14	6	6	4
H_7		1961	97	9	−1	20	9	9	6
H_8		3540	116	−12	3	30	12	12	5
H_9		6253	197	13	−2	37	13	13	8
H_{10}		10654	246	−18	−2	46	18	18	9
$\Gamma_0(N)$		11	22	22	33	33	44	44	55
t		11+	22+	22+11	33+11	33+	44+	44+	55+